Reasoning with minimal belief and negation as failure: algorithms and complexity

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Abstract

We study the computational properties of the propositional fragment of *MBNF*, the logic of minimal belief and negation as failure introduced by Lifschitz, which can be considered as a unifying framework for several nonmonotonic formalisms, including default logic, autoepistemic logic, circumscription, epistemic queries and logic programming. We characterize the complexity and provide algorithms for reasoning in propositional *MBNF*. In particular, we show that skeptical entailment in propositional *MBNF* is Π_3^p-complete, hence, it is harder than reasoning in all the above mentioned propositional formalisms for nonmonotonic reasoning. We also prove the exact correspondence between negation as failure in *MBNF* and negative introspection in Moore's autoepistemic logic.

Introduction

The logic of minimal belief and negation as failure (*MBNF*) (Lifschitz 1994) is one of the most expressive formalisms for nonmonotonic reasoning. Roughly speaking, such a logic is built by adding to first-order logic two distinct modalities, a "minimal belief" modality B and a "negation as failure" modality *not*. The logic thus obtained is characterized in terms of a nice model-theoretic semantics. *MBNF* has been used in order to give a declarative semantics to very general classes of logic programs (Lifschitz & Woo 1992; Lifschitz & Schwarz 1993; Inoue & Sakama 1994), which generalize the stable model semantics of negation as failure in logic programming (Gelfond & Lifschitz 1988; 1990; 1991). Also, *MBNF* can be viewed as an extension of the theory of epistemic queries to databases (Reiter 1990), which deals with the problem of querying a first-order database about its own knowledge.

Due to its ability of expressing many features of nonmonotonic logics (Lifschitz 1994; Lifschitz & Schwarz 1993), *MBNF* is generally considered as a unifying framework for several nonmonotonic formalisms, including default logic, autoepistemic logic, circumscription, epistemic queries and logic programming.

Although several aspects of the logic *MBNF* have been thoroughly investigated (Lifschitz & Schwarz 1993; Chen 1994; Bochman 1995), the existent studies concerning the computational properties of *MBNF* are limited to subclasses of propositional *MBNF* theories (Inoue & Sakama 1994) or to a very restricted subset of the first-order case (Beringer & Schaub 1993). The interest in defining deductive methods for *MBNF* also arises from the fact that such a logic, originally developed as a framework for the comparison of different logical approaches to nonmonotonic reasoning, has recently been considered as an attractive knowledge representation formalism (see e.g. (Donini, Nardi, & Rosati 1997b)).

In this paper we present a computational characterization and provide "optimal" algorithms for deduction in the propositional fragment of *MBNF*. In particular, we show that skeptical reasoning in the propositional fragment of *MBNF* is a Π_3^p-complete problem: hence, it is harder (unless the polynomial hierarchy collapses) than reasoning in all the best known propositional formalisms for nonmonotonic reasoning (Gottlob 1992). Moreover, we study the subclass of flat *MBNF* theories, i.e. propositional *MBNF* theories without nested occurrences of the modalities, showing that in this case entailment is Π_2^p-complete.

We also show that the "negation as failure" modality in *MBNF* exactly corresponds to negative introspection in Moore's autoepistemic logic (Moore 1985). This result implies that the logic *MBNF* can be considered as the "composition" of two epistemic modalities: Halpern and Moses's "minimal knowledge" operator (Halpern & Moses 1985) and Moore's autoepistemic operator.

In the following, we first briefly recall the logic *MBNF*. Then, we study the problem of reasoning in propositional *MBNF* theories. We first tackle the case of flat *MBNF* theories, then we deal with general *MBNF* theories. Finally, we address the relationship between *MBNF* and Moore's autoepistemic logic.

The logic *MBNF*

In this section we briefly recall the logic *MBNF* (Lifschitz 1994). We use $\mathcal{L}$ to denote a fixed propositional language built in the usual way from: (i) an alphabet $\mathcal{A}$ of propositional symbols (atoms); (ii) the symbols true, false; (iii) the propositional connectives $\vee, \wedge, \neg, \supset$. We use the symbol $\vdash$ to denote entailment in propositional logic. We denote with $\mathcal{L}_M$ the modal extension of $\mathcal{L}$ with the modalities B and not. Moreover, we denote with $\mathcal{L}_M^1$ the set of *flat MBNF* formulas, that is the set of formulas from $\mathcal{L}_M$ in which each propositional symbol appears in the scope of exactly one modality, and with $\mathcal{L}_M^S$ the set of *subjective MBNF* formulas, i.e. the subset of formulas from $\mathcal{L}_M$ in which each propositional symbol appears in the scope of at least one modality. We call a modal formula φ from $\mathcal{L}_M$ *positive* (resp. *negative*) if the modality not (resp. B) does not occur in φ. $\mathcal{L}_B$ denotes the set of positive formulas from $\mathcal{L}_M$. We denote with φ^B the positive formula obtained from φ by substituting each occurrence of not with $\neg B$. Analogously, φ^K is the formula obtained from a formula $\varphi \in \mathcal{L}_B$ by substituting each occurrence of B with the modality K.

We now recall the notion of *MBNF* model. An *interpretation* is a set of propositional symbols. Satisfiability of a formula in a structure (I, M_b, M_n), where I is an interpretation (also called initial world) and M_b, M_n are sets of interpretations (worlds), is defined inductively as follows:

1. if φ is an atom, φ is true in (I, M_b, M_n) iff $\varphi \in I$;
2. $\neg\varphi$ is true in (I, M_b, M_n) iff φ is not true in (I, M_b, M_n);
3. $\varphi_1 \wedge \varphi_2$ is true in (I, M_b, M_n) iff φ_1 is true in (I, M_b, M_n) and φ_2 is true in (I, M_b, M_n);
4. $B\varphi$ is true in (I, M_b, M_n) iff, for every $J \in M_b$, φ is true in (J, M_b, M_n);
5. $not\,\varphi$ is true in (I, M_b, M_n) iff there exists $J \in M_n$ such that φ is not true in (J, M_b, M_n).

We also write $(I, M_b, M_n) \models \varphi$ to indicate that φ is true in (I, M_b, M_n).

Definition 1 *A structure (I, M), where $M \neq \emptyset$, is an MBNF* model *of a theory $\Sigma \subseteq \mathcal{L}_M$ iff each formula from Σ is true in (I, M, M) and, for any structure (I', M'), if $M' \supset M$ then at least one formula from Σ is not true in (I', M', M).*

We say that a formula φ is entailed by Σ in *MBNF* (and write $\Sigma \models_{MBNF} \varphi$) iff φ is true in every *MBNF* model of Σ.

From now on, we assume to deal with *finite MBNF* theories Σ, therefore we refer to Σ as a single formula (i.e. the conjunction of all the formulas it contains).

We recall the notion of stability for positive *MBNF* theories, which is analogous to the notion of stable sets in autoepistemic logic.

Definition 2 *A theory $T \subseteq \mathcal{L}_B$ is* stable *if (i) T is closed under propositional consequence, (ii) for every $\varphi \in \mathcal{L}_B$, if $\varphi \in T$ then $B\varphi \in T$, (iii) for every $\varphi \in \mathcal{L}_B$, if $\varphi \notin T$ then $\neg B\varphi \in T$.*

Let $S \subseteq \mathcal{L}$. We denote $ST(S)$ the (unique) stable theory T such that $T \cap \mathcal{L} = \{\varphi | S \vdash \varphi\}$.

Reasoning in propositional *MBNF*

In this section we present algorithms and computational characterizations for entailment in propositional *MBNF*. We refer to (Johnson 1990) for the definition of the complexity classes mentioned in the following.

As in several methods for reasoning in nonmonotonic modal logics (e.g. (Marek & Truszczyński 1993; Eiter & Gottlob 1992)), the techniques we employ are based on a finitary characterization of the preferred models of a theory through subsets of the modal subformulas of the theory. We extend such techniques in order to deal with the preference semantics of *MBNF*. In particular, we are able to provide a method that does not rely on a modal logic theorem prover, but reduces the problem of reasoning in a bimodal logic to (several) reasoning problems in propositional logic.

First, we introduce some preliminary definitions. We say that a propositional symbol p belonging to the alphabet $\mathcal{A}$ *occurs objectively* in $\Sigma \in \mathcal{L}_M$ if there exists an occurrence of p in Σ which is neither in the scope of a B nor in the scope of a not.

Definition 3 *Let $\Sigma \in \mathcal{L}_M$. We denote with $PA(\Sigma)$ the set $\{p : p \in \mathcal{A}$ and p occurs objectively in $\Sigma\}$. Moreover, we call $MA(\Sigma)$ the set of MBNF* atoms *of Σ, which is defined as follows:*

$$MA(\Sigma) = PA(\Sigma) \cup \{B\varphi : B\varphi \text{ is a subformula of } \Sigma\} \cup \{not\,\varphi : not\,\varphi \text{ is a subformula of } \Sigma\}$$

Let $\Sigma \in \mathcal{L}_M$ and let P, N be sets of *MBNF* atoms such that $P \cap N = \emptyset$ and $MA(\Sigma) \cap \mathcal{L}_M^S \subseteq P \cup N \subseteq MA(\Sigma)$. We denote with $\Sigma|_{P,N}$ the formula obtained from Σ by substituting each occurrence in Σ of a formula from P with *true* and each occurrence in Σ of a formula from N with *false*, and simplifying whenever possible. Notice that $\varphi|_{P,N} \in \mathcal{L}$, i.e. $\varphi|_{P,N}$ is a propositional formula.

Let P, N be a partition of $MA(\Sigma)$. Then, we define the following symbols:

$$P_p = P \cap \mathcal{L} \qquad N_p = N \cap \mathcal{L}$$
$$P_m = P \setminus P_p \qquad N_m = N \setminus N_p$$
$$P^+ = \bigwedge_{B\varphi \in P} \varphi|_{P_m,N_m} \qquad N^+ = \bigwedge_{not\,\varphi \in N} \varphi|_{P_m,N_m}$$

P^+ (resp. N^+) corresponds to the *objective knowledge* implied on M (resp. M') in each structure (I, M, M') satisfying the guess on the modal atoms given by (P, N), since in each such structure the propositional formula P^+ is satisfied in each interpretation $J \in M$, i.e. $J \models P^+$, and analogously $J \models N^+$ for each $J \in M'$.

Definition 4 *Let (P, N) be a partition of $MA(\Sigma)$. Then, (P, N) is* consistent *with Σ iff*
(i) the propositional formula $\Sigma|_{P,N}$ is consistent;
(ii) the propositional formula P^+ is consistent;
(iii) the propositional formula N^+ is consistent;
(iv) for each φ such that $B\varphi \in N$, $P^+ \not\vdash \varphi|_{P_m,N_m}$;
(v) for each φ such that not $\varphi \in P$, $N^+ \not\vdash \varphi|_{P_m,N_m}$.

Roughly speaking, the notion of consistency of a partition establishes whether the corresponding guess of the truth values of the *MBNF* atoms of Σ is compatible with Σ and is not self-contradictory.

Let (I, M', M'') be a structure such that Σ is true in (I, M', M''). We say that (I, M', M'') *induces* the partition (P, N) of $MA(\Sigma)$ such that

$$P = \{a : a \in MA(\Sigma) \text{ and } a \text{ is true in } (I, M', M'')\};$$
$$N = MA(\Sigma) \setminus P.$$

Definition 5 *Let $\Sigma \in \mathcal{L}_M$ and let (P, N) be the partition induced by a structure (I, M, M'). We say that (I, M, M') is* maximal with respect to (P, N) *iff $M = \{J : J \models P^+\}$.*

Informally, a model which is maximal with respect to (P, N) is a structure (I, M, M') (which induces (P, N)) in which the propositional formula P^+ exactly characterizes the set of interpretations in M, in the sense that M is the maximal set of interpretations satisfying P^+. Hence, P^+ is the intersection of the objective knowledge of each interpretation in M, i.e. P^+ is the objective knowledge of M.

The following property directly follows from the last definition and from the definition of satisfiability of a formula in a structure.

Lemma 6 *If Σ is true in (I, M, M'), then the partition (P, N) of $MA(\Sigma)$ induced by (I, M, M') is such that (P, N) is consistent with Σ, $M \subseteq \{J : J \models P^+\}$, and $M' \subseteq \{J : J \models N^+\}$.*

The notion of consistent partition provides a finitary characterization of *MBNF* models, as stated by the following theorem.

Theorem 7 *Let $\Sigma \in \mathcal{L}_M$ and let (I, M) be an MBNF model of Σ. Let (P, N) be the partition of $MA(\Sigma)$ induced by (I, M, M). Then, the following conditions hold:*
(i) (P, N) is consistent with Σ;
(ii) for each not $\varphi \in P$, $P^+ \not\vdash \varphi|_{P_m,N_m}$;
(iii) $P^+ \vdash N^+$;
(iv) (I, M, M) is maximal with respect to (P, N).

Informally, the above theorem states that:

1. for *every MBNF* model (I, M) for Σ, there exists a partition (P, N) of $MA(\Sigma)$ consistent with Σ whose corresponding set of interpretations M is $\{J : J \models P^+\}$. Therefore, the search for an *MBNF* model for Σ can be performed by searching among the structures that are maximal with respect to the partitions of $MA(\Sigma)$;

2. the following are *necessary* conditions that a partition (P, N) must satisfy when the maximal structure associated with (P, N) is an *MBNF* model for Σ:
   (a) consistency of (P, N) with Σ: indeed, it is easy to see that, if (P, N) is not consistent with Σ, then for each structure (I, M', M'') which induces (P, N) (and hence for (I, M, M)) it cannot be the case that $(I, M', M'') \models \Sigma$;
   (b) for each *not* $\varphi \in P$, $P^+ \not\vdash \varphi|_{P_m,N_m}$: let (I, M, M) be a structure inducing (P, N), and let *not* $\varphi \in P$. Then, $(I, M, M) \models not\varphi$. This implies $(I, M, M) \models not(\varphi|_{P_m,N_m})$, hence there exists an interpretation $J \in M$ such that $J \models \neg(\varphi|_{P_m,N_m})$. And since $J \models P^+$, it follows that $P^+ \not\vdash \varphi|_{P_m,N_m}$;
   (c) $P^+ \vdash N^+$: if $P^+ \not\vdash N^+$, then there exists at least one interpretation J such that $(I, M \cup \{J\}, M) \models \Sigma$, thus contradicting the hypothesis that (I, M) is an *MBNF* model for Σ.

As for the the entailment problem $\Sigma \models_{MBNF} \varphi$, we point out that the occurrences of *not* in φ are equivalent to occurrences of $\neg B$, since in each *MBNF* model both kinds of modalities in φ are evaluated on the *same* set of interpretations. Therefore, we can transform φ into the unimodal positive formula φ^B.[1]

The following lemma is a straightforward extension of an analogous property proved for stable theories in the case of unimodal logics (Marek & Truszczyński 1993).

Lemma 8 *Let $\Sigma \in \mathcal{L}_M$ and $\varphi \in \mathcal{L}_M$. Let (I, M) be an MBNF model for Σ and let (P, N) be the partition on $MA(\Sigma) \cup PA(\varphi)$ induced by (I, M, M). Then, φ is true in (I, M, M) iff $(\varphi|_{P_p,N_p})^B \in ST(P^+)$.*

The algorithms that we present in the following use the above properties, together with further conditions on partitions (that vary according to the different classes of theories accepted as inputs), in order to find all the *MBNF* models for Σ.

Finally, observe that the fact that a partition of $MA(\Sigma)$ has size linear wrt Σ and the fact that only propositional consistency is involved in the above definition imply that consistency with Σ of a partition can be decided with a

[1] Equivalently, as in Lifschitz's original definition of *MBNF* (Lifschitz 1994), we can restrict query answering to positive *MBNF* formulas only.

Algorithm Flat-Not-Entails(Σ, φ)
Input: formula $\Sigma \in \mathcal{L}_M^1$, formula $\varphi \in \mathcal{L}_M$;
Output: *true* if $\Sigma \not\models_{MBNF} \varphi$, *false* otherwise.
begin
if there exists partition (P, N) of $MA(\Sigma) \cup PA(\varphi)$ **such that**
(a) (P, N) is consistent with Σ **and**
(b) $P^+ \vdash N^+$ **and**
(c) $(\Sigma|_{P_{not}, N})^K \models_{S5} P^+$ **and**
(d) $(\varphi|_{P_p, N_p})^B \notin ST(P^+)$
then return *true*
else return *false*
end

Figure 1: Algorithm Flat-Not-Entails.

polynomial (in the size of Σ) number of calls to an NP-oracle.

Flat *MBNF* theories

We now study the complexity of entailment for flat *MBNF* theories.

In the case of flat *MBNF* theories and subjective queries, entailment is Π_2^p-complete: membership in the class Π_2^p is a straightforward consequence of a property shown in (Schwarz & Truszczyński 1994), while Π_2^p-hardness follows from the existence of a polynomial-time embedding of propositional default theories into flat *MBNF* theories (Lifschitz 1994; Gottlob 1992). Therefore, the following property holds.

Proposition 9 *Let $\Sigma \in \mathcal{L}_M^1$ and let $\varphi \in \mathcal{L}_M^S$. Then, the problem of deciding whether $\Sigma \models_{MBNF} \varphi$ is Π_2^p-complete.*

It is also known that the decision procedure for computing entailment in the logic $\mathsf{S4F}_{MDD}$ presented in (Marek & Truszczyński 1993) can be employed for computing entailment for flat *MBNF* theories, by a simple translation of *MBNF* formulas into unimodal formulas of $\mathsf{S4F}_{MDD}$ (Schwarz & Truszczyński 1994).

We now study a more general problem, that is entailment $\Sigma \models_{MBNF} \varphi$ in the case $\Sigma \in \mathcal{L}_M^1$ and $\varphi \in \mathcal{L}_M$. We exhibit the algorithm Flat-Not-Entails, reported in Figure 1, for computing entailment in such a case. In the algorithm, P_{not} denotes the set $\{not\,\varphi : not\,\varphi \in P\}$.

Informally, correctness of the algorithm Flat-Not-Entails is established by the fact that, if $\Sigma \in \mathcal{L}_M^1$, then (a),(b),(c) are necessary and sufficient conditions on a partition (P, N) in order to establish whether it identifies an *MBNF* model for Σ. In particular, condition (c) states that P^+ must be a consequence of $(\Sigma|_{P_{not}, N})^K$ in modal logic S5[2], since it can be shown that if $(\Sigma|_{P_{not}, N})^K \not\models_{S5} P^+$, then the guess on the modal atoms of the form $B\varphi$ in P is not minimal. We try to illustrate this fact through the following example.

Suppose $\Sigma = Ba \vee B(a \wedge b)$, and suppose $P = \{Ba, B(a \wedge b)\}$, $N = \emptyset$. Then, $(\Sigma|_{P_{not}, N})^K = Ka \vee K(a \wedge b)$ and $P^+ = a \wedge b$, hence $(\Sigma|_{P_{not}, N})^K \not\models_{S5} P^+$. Let (I, M, M) be maximal wrt (P, N); hence, $M = \{J : J \models a \wedge b\}$. Now let $M' = \{J : J \models a\}$: it is easy to see that $(I, M', M) \models \Sigma$, and since $M' \supset M$, (I, M) is not an *MBNF* model for Σ.

Theorem 10 *Let $\Sigma \in \mathcal{L}_M^1$ and $\varphi \in \mathcal{L}_M$. Then, Flat-Not-Entails(Σ, φ) returns true iff $\Sigma \not\models_{MBNF} \varphi$.*

The non-deterministic algorithm Flat-Not-Entails runs in Σ_2^p, since it can be shown that each of the conditions in the **if** statement can be computed through a number of calls to an oracle for the propositional validity problem which is polynomial in the size of $(\Sigma \cup \varphi)$. In particular, the following property holds (see (Marek & Truszczyński 1993, Theorem 9.40)).

Proposition 11 *Let $\Sigma \in \mathcal{L}_M^1$, $\varphi \in \mathcal{L}_M$, and let (P, N) be a partition of $MA(\Sigma) \cup PA(\varphi)$ consistent with Σ. Then, membership of the positive formula $(\varphi|_{P_p, N_p})^B$ in the stable theory $ST(P^+)$ can be computed by an algorithm performing a polynomial (in the size of φ) number of calls to an oracle for the propositional validity problem.*

Therefore, the following property holds.

Theorem 12 *Let $\Sigma \in \mathcal{L}_M^1$ and let $\varphi \in \mathcal{L}_M$. Then, the problem of deciding whether $\Sigma \models_{MBNF} \varphi$ is Π_2^p-complete.*

Hence, the algorithm Flat-Not-Entails is "optimal" in the sense that it matches the lower bound for the entailment problem.

We remark the fact that the subset of flat *MBNF* theories in conjunctive normal form can be seen as a further extension of the framework of *generalized logic programming* introduced in (Inoue & Sakama 1994), which in turn is an extension of the disjunctive logic programming framework under the stable model semantics presented in (Gelfond & Lifschitz 1991). Roughly speaking, flat *MBNF* theories in conjunctive normal form correspond to rules of generalized logic programs in which propositional formulas (instead of literals) are allowed as goals. The above computational characterization implies that such an extension of the framework of logic programming under the stable model semantics does not affect the worst-case complexity of the entailment problem, which is Π_2^p-complete just like entailment in logic programs with disjunction under the stable model semantics (Eiter & Gottlob 1995).

General *MBNF* theories

We start by establishing a lower bound for reasoning in general *MBNF* theories. To this end, we exploit the correspondence between *MBNF* and Halpern and Moses's logic of minimal knowledge (Halpern & Moses 1985), also known

[2]We denote with K the modal operator used in S5.

Algorithm *MBNF*-Not-Entails(Σ, φ)
Input: formula $\Sigma \in \mathcal{L}_M$, formula $\varphi \in \mathcal{L}_M$;
Output: *true* if $\Sigma \not\models_{MBNF} \varphi$, *false* otherwise.
begin
if there exists partition (P, N) of $MA(\Sigma) \cup PA(\varphi)$
such that
  (P, N) is consistent with Σ **and**
  $P^+ \vdash N^+$ **and**
  $(\varphi|_{P_p,N_p})^B \notin ST(P^+)$ **and**
  for each partition (P', N') of $MA(\Sigma')$,
  where $\Sigma' = \Sigma \wedge \neg not(P^+)$
    (a) (P', N') is not consistent with Σ' **or**
    (b) $P^+ \not\vdash P'^+$ **or**
    (c) $P'^+ \vdash P^+$ **or**
    (d) $P^+ \not\vdash N'^+$
then return *true*
else return *false*
end

Figure 2: Algorithm *MBNF*-Not-Entails.

as ground nonmonotonic modal logic $S5_G$ (Donini, Nardi, & Rosati 1997). Indeed, it is easy to see that, for *positive* subjective *MBNF* theories, there is a one-to-one correspondence between *MBNF* models of a theory Σ and $S5_G$ models of the theory Σ^K.

Proposition 13 *Let* $\Sigma \in \mathcal{L}_M^S$. *Then,* (I, M) *is an MBNF model for* Σ *iff* M *is an* $S5_G$ *model for* Σ^K.

Therefore, the interpretation of the *MBNF* operator B exactly corresponds to the interpretation of the modality K in $S5_G$.

As shown in (Donini, Nardi, & Rosati 1997), entailment in $S5_G$ is Π_3^p-complete. Therefore, for subjective (and hence for general) *MBNF* theories, entailment is Π_3^p-hard.

Lemma 14 *Let* $\Sigma \in \mathcal{L}_M^S$ *and let* $\varphi \in \mathcal{L}_M$. *Then, the problem of deciding whether* $\Sigma \models_{MBNF} \varphi$ *is* Π_3^p*-hard.*

Then, we show that entailment for general *MBNF* theories is a problem belonging to the class Π_3^p. To this end, we define the non-deterministic algorithm reported in Figure 2.

Such an algorithm makes use of the finitary characterization of *MBNF* models given in Theorem 7 and of an analogous finitary characterization, in terms of partitions of $MA(\Sigma) \wedge \neg not(P^+)$, of all the models relevant for establishing whether a partition (P, N) of $MA(\Sigma)$ consistent with Σ identifies an *MBNF* model. Informally, by Theorem 7, each partition consistent with Σ and such that $P^+ \vdash N^+$ identifies a unique *MBNF* model, i.e. the structure (I, M) such that (I, M, M) is maximal with respect to (P, N). Now, it can be shown that, in order to establish the existence of a structure (J, M, M') such that Σ is true in (J, M, M') (which implies that (I, M) is not an *MBNF* model of Σ), it is sufficient to check for the presence, among all the consistent partitions of $\Sigma \wedge \neg not(P^+)$, of a partition that falsifies each of the conditions (a),(b),(c),(d) in the algorithm.

Theorem 15 *Let* $\Sigma \in \mathcal{L}_M$, $\varphi \in \mathcal{L}_M$. *Then, MBNF-Not-Entails(Σ, φ) returns true iff* $\Sigma \not\models_{MBNF} \varphi$.

It can be shown that the above non-deterministic algorithm runs in Σ_3^p, since falsity of the **for each** condition can be checked in Σ_2^p. Therefore, we obtain the following computational characterization of the entailment problem for propositional *MBNF* theories.

Theorem 16 *Let* $\Sigma \in \mathcal{L}_M$ *and let* $\varphi \in \mathcal{L}_M$. *Then, the problem of deciding whether* $\Sigma \models_{MBNF} \varphi$ *is* Π_3^p*-complete.*

We point out the fact that the algorithm *MBNF*-Not-Entails does not rely on a theorem prover for a modal logic: thus, "modal reasoning" is not actually needed for reasoning in *MBNF*. This is an interesting feature that *MBNF* shares with other nonmonotonic modal formalisms, like autoepistemic logic (Moore 1985) or the autoepistemic logic of knowledge (Schwarz 1991).

The previous analysis also allows for a computational characterization of the logic *MKNF* (Lifschitz 1991), which is a slight modification of *MBNF*. Indeed, for each theory $\Sigma \subseteq \mathcal{L}_M$, M is an *MKNF* model of Σ iff (I, M) is an *MBNF* model of the subjective theory $\{B\varphi : \varphi \in \Sigma\}$. Therefore, the following property holds.

Corollary 17 *Entailment in propositional MKNF is* Π_3^p*-complete.*

Finally, the previous theorem provides a computational characterization of the logic of grounded knowledge and justified assumption GK (Lin & Shoham 1992). In fact, the logic GK can be considered as a syntactic variant of the propositional fragment of *MKNF*. Therefore, skeptical entailment in GK is Π_3^p-complete.

Remark. The fact that skeptical reasoning in the propositional fragment of *MBNF* is a Π_3^p-complete problem relates from the computational viewpoint propositional *MBNF* with ground nonmonotonic modal logics (Eiter & Gottlob 1992; Donini, Nardi, & Rosati 1997). Notably, such formalisms share with *MBNF* the interpretation in terms of minimal knowledge (or minimal belief) of the modality B; in particular, as mentioned, the propositional fragment of *MBNF* can be considered as built upon $S5_G$ by adding a second modality *not*. Therefore, it turns out that, in the propositional case, adding a "negation by default" modality to the S5 logic of minimal knowledge does not increase the computational complexity of reasoning. In other words, the minimal belief operator adds more complexity to the deduction problem than the negation by default operator.

Relating *MBNF* with autoepistemic logic

Finally, we show that negation as failure in *MBNF* exactly corresponds to negative introspection in Moore's autoepistemic logic (*AEL*) (Moore 1985). In order to keep notation to a minimum, we change the language of *AEL*, using the modality B instead of L. Thus, in the following an autoepistemic formula in *AEL* is a formula from $\mathcal{L}_B$. Also, with the term *AEL model* for Σ we will refer to an S5 model whose set of theorems corresponds to a consistent stable expansion for Σ in *AEL*: we will denote such a model with the set of interpretations it contains.

We first define the translation $tr_M(\cdot)$ of modal theories from *AEL* to *MBNF* theories.

Definition 18 *Let $\varphi \in \mathcal{L}_K$. Then, $tr_M(\varphi)$ is the MBNF formula $B(\varphi')$, where φ' is obtained from φ by substituting each occurrence of B with $\neg not$. Moreover, if $\Sigma \subseteq \mathcal{L}_K$, then $tr_M(\Sigma)$ denotes the MBNF theory $\{tr_M(\varphi) | \varphi \in \Sigma\}$.*

The translation $tr_M(\cdot)$ embeds *AEL* theories into *MBNF*, as stated by the following theorem.

Theorem 19 *Let $\Sigma \subseteq \mathcal{L}_K$. Then, M is an AEL model for Σ iff, for any interpretation I, (I, M) is an MBNF model of $tr_M(\Sigma)$.*

The previous theorem implies that the modality *not* has in *MBNF* the *same interpretation* of the modal operator of autoepistemic logic. This property extends previous results relating *MBNF* with *AEL* (Lin & Shoham 1992; Lifschitz & Schwarz 1993; Chen 1994), and has many consequences both in the logic programming framework and in nonmonotonic reasoning. In particular, it allows us to interpret the logic *MBNF* (and *MKNF*) as the "composition" of two nonmonotonic modal formalisms: Halpern and Moses's logic of minimal knowledge $S5_G$ and Moore's *AEL*. This result further clarifies the relationship between several different nonmonotonic formalisms, thus strengthening the idea that *MBNF* is a unifying framework for nonmonotonic reasoning.

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