A DESIGN METHOD FOR RELAXATION LABELING APPLICATIONS

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ABSTRACT

A summary of mathematical results developing a theory of consistency in ambiguous labelings is presented. This theory allows the relaxation labeling algorithm, introduced in [Rosenfeld, Hummel, Zucker, 1976], to be interpreted as a method for finding consistent labelings, and allows specific applications to be tailored in accordance with intended design goals. We discuss, with a couple of examples, a design methodology for using this theory for practical applications.

I FOUNDATIONS OF RELAXATION LABELING

We begin by presenting a succinct summary of the theory developed in [Hummel and Zucker, 1983]. For details and mathematical proofs, the reader is referred to the references.

Let $a_1,\dots,a_n$ denote distinct objects, and $\{\lambda_1,\dots,\lambda_m\}$ be a set of possible labels. The goal is to assign one label to each object. In most practical systems, measurements are made to describe each object a_i, and a most probable label is assigned independent of the labels assigned to neighboring objects. Relaxation labeling is an iterative method for incorporating context and local consistency in weighted labeling assignments. In this model, a nonnegative weight $p_i(\ell)$ is assigned for each label λ_ℓ at every object a_i, normalized by the conditions $\sum_{\ell=1}^{m} p_i(\ell) = 1$, $i = 1,\dots,n$. The <u>weighted labeling assignment</u> $p_i(\ell)$ denotes a confidence level for the assignment of label λ_ℓ at object a_i. The concatenation of the weighted assignment values comprise the <u>assignment vector</u> $\bar{p} = (\bar{p}_1,\dots,\bar{p}_n)$, $\bar{p}_i = (p_i(1),\dots,p_i(m))$. The space of possible assignment vectors $\bar{p}$ is the <u>assignment space</u> K. An <u>unambiguous assignment</u> $\bar{p} \in K$ is an assignment vector satisfying $p_i(\ell) = 0$ or 1 for all i and ℓ. Because of the normalization condition, an unambigous assignment gives a weight of 1 to exactly one label at each object. We denote the set of all unambiguous assignments by K*, and note that K is the simplex formed by the convex hull of K*.

To apply the relaxation labeling method, constraints between neighboring labels are specified by a set of weighted preferences, given by support functions $s_i(\ell;\bar{p})$, which are functions of the labeling assignment $\bar{p}$. A separate function is given for each label at each object. The support $s_i(\ell,\bar{p})$ can be positive or negative, and denotes the support which the current mix of weighted assignments lend to the proposition that object a_i is label λ_ℓ. In nearly all applications to date, the support functions are linear in $\bar{p}$:

$$s_i(\ell;\bar{p}) = \sum_j \sum_{\ell'} r_{ij}(\ell,\ell')\, p_j(\ell'),$$

and depend principally on assignments $p_j(\ell')$ at objects a_j near object a_i.

It is of course essential to know where the support functions come from. In first stating the theory, it is enough to suppose that the $s_i(\ell;\bar{p})$'s are "God-given". However, the principal task confronting the designer of a relaxation labeling applications is the definition of formulae for computing support functions. In Section III, we show how the distinction between consistent and inconsistent labelings can be used to constrain the choice of support functions.

We must first make the distinction precise. By using support functions, we can extend the usual notion of consistency (as defined, for example, in [Mackworth, 1977] and [Haralick and Shapiro 1979]) to reflect a quantitative system of preferences. We state here the definition for unambiguous labelings, and refer the reader to [Hummel and Zucker, 1983] for a discussion and the weighted labeling assignment version.

Suppose $\bar{p} \in K^*$, which is to say that $\bar{p}$ assigns to each object a_i a single label λ_{ℓ_i}, signified by $p_i(\ell_i) = 1$. The unambiguous labeling $\bar{p}$ is <u>consistent</u> iff

$$s_i(\ell_i;\bar{p}) \geq s_i(\ell;\bar{p}) \quad \text{for all } \ell \text{ and } i.$$

That is, at each object the maximum support value is attained for the label actually assigned to the object.

The relaxation labeling algorithm, defined precisely in [Hummel and Zucker, 1983], is an iterative process for updating initial weighted labeling assignments in the assignment vector $\bar{p}^0$ to achieve a consistent assignment. Heuristically, the idea is to continuously increase $p_i(\ell)$ if $s_i(\ell;\bar{p})$ is positve, and to decrease $p_i(\ell)$ if $s_i(\ell;\bar{p})$ is negative, subject to the constraint that $\bar{p}$ remain in the assignment space K. On the k^{th} iteration of the algorithm, the current assignment vector $\bar{p}^k$ is updated by first computing the updating vector $\bar{q}^k \in (\mathbb{R}^n)^m$, whose components are given by $q_i^k(\lambda) = s_i(\ell;\bar{p}^k)$, then projecting $\bar{q}^k$ onto the tangent set to the space K at the point $\bar{p}^k$ to yield a tangent direction $\bar{u}^k$, and finally stepping in the direction $\bar{u}^k$ by setting $\bar{p}^{k+1} = \bar{p}^k + \alpha\bar{u}^k$, where α is a sufficiently small positive constant. This process is repeated until convergence. The projection operator required to obtain $\bar{u}^k$ from $\bar{q}^k$ is described in [Mohammed, Hummel, Zucker, 1983].

The formulation of relaxation labeling outlined above leads to a number of theorems, proved in [Hummel and Zucker, 1983]. Two results of particular relevance here:

I. If the relaxation labeling process stops at $\bar{p}$, then $\bar{p}$ is a consistent labeling.

II. A strictly consistent unambiguous labeling ($s_i(\ell_i;\bar{p}) > s_i(\ell;\bar{p})$ for $\ell \neq \ell_i$, all i) is a local attractor of the relaxation labeling algorithm.

II DESIGNING SUPPORT FUNCTIONS

Suppose that objects and label sets have been identified, and that formulae for support functions are required. The following method is suggested.

Certain patterns of unambiguous labelings can be identified as consistent labelings. Suppose, for example, that an unambiguous assignment $i \to \ell_i$, denoted succinctly by $\bar{\ell} = (\ell_1 \ldots \ell_n)$, is to be viewed as consistent. Let $\bar{p}$ be the corresponding unambiguous assignment vector. Then we want the inequalities $s_i(\ell_i;\bar{p}) \geq s_i(\ell;\bar{p})$ to be satisfied. If the support functions are to be designed as linear functions of $\bar{p}$, these conditions can be written as

$$\sum_j r_{ij}(\ell_i,\ell_j) \geq \sum_j r_{ij}(\ell,\ell_j), \text{ all } \ell,\ i.$$

Suppose that $\bar{\ell}^1,\ldots,\bar{\ell}^N$ are N distinct patterns of labelings which are deemed to be consistent. Then we want

$$\sum_j r_{ij}(\ell_i^k,\ell_j^k) \geq \sum_j r_{ij}(\ell,\ell_j^k)$$

to be satisfied for all ℓ, all i, for k = 1,...,N. These conditions may constitute a large number of linear inequalities in the set of variables $r_{ij}(\ell,\ell')$.

The system of inequalities may have no nontrivial solution, in which case it is impossible to design linear support functions with $\bar{\ell}^1,\ldots,\bar{\ell}^N$ as consistent labeling patterns. However, if the system has a nonempty nontrivial solution set, then any assignment of values to the $r_{ij}(\ell,\ell')$'s satisfying the inequalities is called a feasible solution. In this case, linear programming methods (such as the simplex method) can be used to find feasible solutions. If the coefficients are chosen from the interior of the set of feasible solutions, then the solution is called strictly feasible, and Result II from the previous section can be used to show the given patterns $\bar{\ell}^1\ldots\bar{\ell}^N$ will then correspond to unambiguous labelings which are local attractors of the relaxation labeling process.

It may well happen that a strictly feasible solution for the $r_{ij}(\ell,\ell')$'s will yield other consistent unambiguous labelings not represented in the design pattern set. This may be undesirable, and will require a search for a feasible solution which minimizes the problem of spurious consistent labelings. The second example given below illustrates a method for accomplishing this search.

IV EXAMPLES

Suppose that the graph of objects is given by a hexagonal grid, so that each object is equidistant from its six neighbors. Consider the simplest case of two labels. We suppose that the following local patterns are consistent:

```
 1 1      2 2      1 2      2 1
1 1 1    2 2 2    1 1 2    2 2 1 .
 1 1      2 2      1 1      2 2
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Further, we assume that the relationship of consistency is "isotropic," that is, a rotation of a consistent labeling is consistent. Since the labels 1 and 2 are treated symmetrically, we can also assume that $r_{ij}(1,1) = r_{ij}(2,2)$ and $r_{ij}(1,2) = r_{ij}(2,1)$. Finally, we assume that the r_{ii}'s are zero, and the r_{ij}'s are independent of i and j as long as i and j are distinct neighboring objects. Thus only two parameters are sought: $a = r_{ij}(1,1)$ and $b = r_{ij}(1,2)$, for i and j distinct neighbors. Applying the conditions for strict consistency to each of the patterns listed above leads to the single condition $a > b$.

What other unambiguous local patterns are consistent in this scheme? It is not hard to show that (for $a > b$) a local pattern with a central object labeled "1" is strictly consistent if the number of the six neighbors with a "1" label, n_1, is greater than the number of neighbors with a "2" label, n_2. Similarly, "2" is consistent for the central object if $n_2 > n_1$. A global unambiguous labeling will be consistent if every object is labeled with the majority label as voted by the six neighbors. Since this condition holds at every object, strictly consistent labelings consist of strips of 1's and 2's with straight parallel interfaces between the regions.

We next present a slightly more complicated example. However, practical situations will generally be much more complex than either of our examples. This time, consider a hexagonal array of objects with three labels. The labels "1" and "2" are regarded as "region types", and label "3" denotes "edge between 1's and 2's." A pattern of constant 1's or 2's, and a region of 1's separated from 2's by a line of 3's are consistent labelings:

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  1 1       2 2       3 2
 1 1 1     2 2 2     1 3 2 .
  1 1       2 2       1 3
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We must also regard

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  1 1     and     2 3
 1 1 3           2 2 3
  1 1             2 2
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as consistent. As before, we treat labels 1 and 2 symmetrically, and assume isotropy.

Five parameters arise: $a = r(1,1) = r(2,2)$, $b = r(1,2) = r(2,1)$, $c = r(3,3)$, $d = r(1,3) = r(2,3)$, and $e = r(3,1) = r(3,2)$. Here i and j are suppressed since objects i and j can be only distinct pairs of neighbors. Note also that the r's are not necessarily symmetric -- if $d \neq e$, then regions can influence borders differently than borders influence regions.

Each consistent pattern yields two inequalities to constrain the five parameters. From the constant patterns, we deduce that $a > b$ and $a > e$. From the pattern with a central line of 3's, $4e + 2c > 2a + 2b + 2d$, and from the remaining pattern we obtain $2a + d > 2e + c$. Combining, we have

$$a > b, \quad a > e, \quad \text{and}$$

$$(a-e) + (b-e) < c-d < 2(a-e).$$

It is easy to see that feasible solutions of these equations exist and can be readily constructed. In particular, choose any positive value for a', then choose $b' < a'$, and e arbitrary. Set $a = a' + e$, $b = b' + e$, and finally choose a value for $c - d$ between $a' + b'$ and $2a'$. The values for c and d can be selected so that the difference is the specified value.

The designated patterns, and all their rotations, will be strictly consistent under the compatibilities of any feasible solution. We will now try to select a feasible solution which gives rise to as few other consistent labeling patterns as possible.

Let n_k denote the number of neighbors of a central point of a hexagonal cell having label "k", for $k = 1,2$, or 3. The label "1" at the central object is part of a consistent labeling if its support, $an_1 + bn_2 + dn_3$, is greater than the support for the labels "2" and "3", i.e., $dn_1 + an_2 + dn_3$ and $en_1 + en_2 + cn_3$. From the two inequalities, and the fact that $n_1 + n_2 + n_3 = 6$, we deduce that "1" is consistent if

$$n_1 > n_2 \text{ and } [(c-d)+(b-e)]n_3 < (a-b)n_1 + 6(b-e).$$

Similarly, "3" is a consistent label for the central object of a local pattern if

$$[(c-d)+(b-e)]n_3 > (a-b)\cdot\max(n_1,n_2) + 6(b-e).$$

Let us arbitrarily choose $a = 1$ and $b = -1$. Then $e = 0$ makes sense since "3" and "1" can co-occur. Having chosen a, b, and e, then $0 < c-d < 2$. Suppose we choose $c-d = 1$. Then applying the conditions above, a central "1" is consistent if $n_1 > n_2$ and $n_1 > 3$. for a "3" to be consistent, it suffices to have $\max(n_1,n_2) < 3$. Thus a pattern with a "1" or "2" label in the center is consistent if there are four or more of the same label type in the neighborhood. A central "3" is consistent as long as there are two or fewer "1"'s and two or fewer "2"'s in the neighborhood.

This seems reasonable. However, note that the pattern

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 1 2
1 x 2
 1 1
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is consistent with $x = 1$ under the above choice of values. We would prefer the support for $x = 3$ to be higher than the support for $x = 1$ to justify the interpretation of label "3" as "edge". Thus we would like $6e > 4a + 2b$. Having chosen $a = 1$, $b = -1$, we now see that $e > 1/3$ is desirable. Let $e = 2/3$, whence $-2/3 < c-d < 4/3$. By varying the value of the one parameter c-d, different behavior of the patterns of consistency can be selected. For example, suppose we select $c-d = 1/2$. Then a case analysis yields:

A label "1" is consistent only if there are four or more "1" labels among the six neighbors, and no "2" labels;

A "3" label is consistent if there are three or more "3" labels among the neighbors, or if there are four of label type "1" or "2", and at least one of the other type.

This example illustrates how an initial assignment of values to the $r_{ij}(\ell,\ell')$'s obtained as a feasible solution can be refined to give more desirable behavior by the addition of a constraint. In this case, the additional inequality arises when we decide to reject a spurious consistent pattern, and the statement that a particular label hsould have greater support than the otherwise consistent label.

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